Large scale structure formation
with the
Schrödinger method

Cora Uhlemann
Arnold Sommerfeld Center, LMU
& Excellence Cluster Universe
Advisor: Stefan Hofmann
in collaboration with
Michael Kopp, University of Cyprus

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-13.8 billion years: nearly uniform initial state

today: rich structures in cosmic web
-13.8 billion years: nearly uniform initial state

Inflation
• established `boring` initial conditions
  • quantum fluctuations get amplified
  • primordial plasma cools $\rightarrow$ recombination $\rightarrow$ CMB

Structure formation
• hierarchical
• tiny over-densities act as seeds
  • congregation via gravitational instability
  • collapse into bound structures

Large scale structure: Cold Dark Matter
• linear regime
  ✓ analytically understood
• nonlinear stage
  ?! N-body simulations inevitable

today: rich structures in cosmic web
Describing Cold Dark Matter with the Schrödinger method
Describing Cold Dark Matter

Phase space distribution function $f(t,x,p)$
- describes number density & distribution of momenta $p$

**Theoretical expectation**

**Numerical realization**

Pueblas & Scoccimarro (2009, PRD 80)

Widrow (1997, PRD 55)
Describing Cold Dark Matter

phase space distribution function \( f(t,x,p) \)
- **N-body**: non-relativistic, only gravitationally
- **continuous**: ensemble average, no collisions

\[
f_N = \sum_i \delta_D(x - x_i) \delta_D(p - p_i)
\]

Vlasov - Poisson equation

\[
\partial_\tau f(x, p, \tau) = -\frac{p}{am} \nabla_x f + am \nabla_x V \nabla_p f
\]

gravitational potential

\[
\Delta V(x, \tau) = \frac{4\pi Gm}{a} (n(x, \tau) - \langle n \rangle)
\]

integro

number density

\[
n = \int d^3p \ f
\]

Solving is hard!

have to choose a special ansatz
for phase space distribution \( f(x,p) \)
Describing Cold Dark Matter

phase space distribution function \( f(t,x,p) \)
- **N-body**: non-relativistic, only gravitationally
- **continuous**: ensemble average, no collisions

\[
\partial_\tau f(x,p,\tau) = -\frac{p}{am} \nabla_x f + am \nabla_x V \nabla_p f
\]

**Vlasov - Poisson equation**

\[
\nabla \cdot \Delta V(x,\tau) = \frac{4\pi G m}{a} (n(x,\tau) - \langle n \rangle)
\]

**Hierarchy of Moments**

\[
M^{(n)}(x) = \int d^3p \ p_{i_1} \ldots p_{i_n} f
\]

- density \( n(x) \): \( M^{(0)} = n(x) \), velocity \( v(x) \): \( M^{(1)} = n v(x) \)
- velocity dispersion \( \sigma(x) \): \( M^{(2)} = n (v v + \sigma(x)), \ldots \)

\[
\partial_t M^{(n)} = -\frac{1}{a^2 m} \nabla \cdot M^{(n+1)} - m \nabla V \cdot M^{(n-1)}
\]

infinite coupled hierarchy
Dust model

**dust model**
- only consistent truncation of hierarchy
- pressureless fluid: density and velocity

\[ f_d(x, p, \tau) = n(x, \tau) \delta_D^3 (p - \nabla \phi(x, \tau)) \]

**Continuity**
\[ \partial_\tau n = -\frac{1}{am} \nabla (n \nabla \phi) \]

**Euler**
\[ \partial_\tau \phi = -\frac{1}{2am} (\nabla \phi)^2 - amV \]

- limited to **single-stream**
- no velocity dispersion, …
- shell-crossing singularities
- no virialization
Schrödinger method at a glance

**WANTED**
N-body double $f_N$

$\sigma_x \ll x_{\text{typ}}$

$\sigma_p \ll p_{\text{typ}}$

**coarse grained Vlasov** $f_{\text{cg}}$

\[ \begin{align*}
\text{time-evolution} \\
\text{shell-crossing}
\end{align*} \]

$\rightarrow$

**dust** $f_d$

$\rightarrow$

**coarse grained Wigner** $f_{\text{cg}}$

$\leftarrow$

Takahashi (1989, PTP 98)

$\rightarrow$

**Wigner** $f_W$
Schrödinger method

Schrödinger method
- Coarse-grained Wigner function, constructed from self-gravitating field

\[ f_{cw}(x, p) = \int \frac{d^3x' d^3p'}{(\pi \sigma_x \sigma_p)^3} \exp \left[ -\frac{(x - x')^2}{2\sigma_x^2} - \frac{(p - p')^2}{2\sigma_p^2} \right] \]

degrees of freedom
- 2: amplitude n & phase \( \phi \)

parameters
- coarse-graining \( \sigma_x, \sigma_p \)
  - fundamental resolution \( \sigma_x \sigma_p \gtrsim \hbar / 2 \)
- Schrödinger scale \( \hbar \)
  - degree of restriction
  - dust as special case

\[ \psi = \sqrt{n} \exp \left( \frac{i}{\hbar} \phi \right) \]

Schrödinger - Poisson equation

\[ i\hbar \partial_t \psi = \left[ -\frac{\hbar^2}{2am} \Delta + mV \right] \psi \]

\[ \Delta V = \frac{4\pi G \rho_0}{a} (|\psi|^2 - 1) \]

Continuity
\[ \partial_\tau n = -\frac{1}{am} \nabla (n \nabla \phi) \]

Euler
\[ \partial_\tau \phi = -\frac{1}{2am} (\nabla \phi)^2 - amV + \frac{\hbar^2}{2am} \left( \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) \]

quantum potential
Features of Schrödinger Method

Multi-streaming

❌ dust model: fails at shell-crossing
✅ Schrödinger method: can go beyond shell-crossing

blue S contours: Schrödinger method
red dotted Z line: Zeldovich solution (dust model)

Virialization

❌ even in extended models: no virialization
✅ Schrödinger method: bound structures like halos
Features of Schrödinger Method

- Prevention of shell-crossing singularities
- Occurrence of phase jumps
- \( \psi = \sqrt{n} e^{i\phi/h} \) free of pathologies
Features of Schrödinger Method

Schrödinger method

- **Coarse-grained Wigner function**, constructed from self-gravitating field

\[ f_{\text{W}}(x, p) = \int \frac{d^3 x' d^3 p'}{(\pi \sigma_x \sigma_p)^3} \exp \left[ -\frac{(x - x')^2}{2\sigma_x^2} - \frac{(p - p')^2}{2\sigma_p^2} \right] \int \frac{d^3 \tilde{x}}{(2\pi \hbar)^3} \exp \left[ \frac{i}{\hbar} p' \cdot \tilde{x} \right] \psi(x' - \tilde{x}) \bar{\psi}(x' + \tilde{x}) \]

Cumulants

- lowest two: macroscopic density & velocity

\[ \bar{n}(x) = \exp \left[ \frac{1}{2} \sigma_x^2 \Delta \right] n(x) \quad \bar{v}(x) = \frac{1}{am\bar{n}(x)} \exp \left[ \frac{1}{2} \sigma_x^2 \Delta \right] (n \nabla \phi)(x) \]

- higher cumulants given self-consistently

\[ C^{(0)} = \ln n, \quad C^{(1)} = \nabla \phi \]

\[ C^{(n+2)} = -\frac{\hbar^2}{4} \nabla \nabla C^{(n)} \quad \text{from Wigner} \]

add coarse-graining to determine the moments

**closure of hierarchy**

CU, Kopp & Haugg (2014, PRD 90, 023517)
Multi-streaming

- higher cumulants encode multi-streaming effects
- during shell-crossing: higher moments sourced dynamically

Schrödinger method: cumulants at $x = -0.5$ Mpc: all equally important after shell crossing
Schrödinger method at a glance

**WANTED**

N-body double $f_N$

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$V$-body double $f_N$

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Coarse grained Vlasov $f_{cg}$

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Coarse grained Wigner $f_{cg}$

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**dust** $f_d$

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$W$-body double $f_W$

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Virialization

- **FAILS** even in extended models
- **SUCCESS** in bound structures

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**Features of Schrödinger Method**

- **Beyond shell-crossing**
- **No virialization**

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Virial density $C_{1,2,3}$

- $C_1 = 0.0$
- $C_2 = 0.5$
- $C_3 = 1.0$
- $C_4 = 1.5$
- $C_5 = 2.0$
- $C_6 = 2.5$

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After shell-crossing: all cumulants important
**Schrödinger method**
- models CDM using a self-gravitating scalar field
- analytical tool to access nonlinear stage of structure formation
  - describes multi-streaming
  - allows for virialization

**Future research**
- understand universal density profiles of halos (NFW)
  - search stationary solutions of gravitational collapse
- consider a flow of time or phase-space resolution $\hbar$
  - possible interpretation in terms of phase transition
- DM models: wavelike (axion), warm & (non-)relativistic neutrinos